ORDER PARAMETER EVOLUTION IN SCALAR QFT: RENORMALIZATION GROUP RESUMMATION OF SECULAR TERMS

H. J. de Vega and J. F. J. Salgado*

L.P.T.H.E.†, Universités Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII), Tour 16, 1^{er} étage, 4 Place Jussieu, 75252 Paris Cedex 05, FRANCE

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The quantum evolution equations for the field expectation value are analytically solved to cubic order in the field amplitude and to one-loop level in the $\lambda\phi^4$ model. We adapt and use the renormalization group (RG) method for such non-linear and non-local equations. The time dependence of the field expectation value is explicitly derived integrating the RG equations. It is shown that the field amplitude for late times approaches a finite limit as $O(t^{-3/2})$. This limiting value is expressed as a function of the initial field amplitude.

I. INTRODUCTION AND MOTIVATIONS

An important research activity develops presently on the non-equilibrium dynamics of quantum field theory with multiple physical motivations [1–5,8]. They arise in the reheating of inflationary universes, the eventual formation of disordered chiral condensates [6], the understanding the hadronization stage of the quark-gluon plasma [9] as well as trying to understand out of equilibrium particle production in strong electromagnetic fields and in heavy ion collisions [10,11].

A common feature in all such phenomena is the presence of quantum fields with large amplitudes. That is, they imply the quantum field dynamics when the energy density is **high**. That is to say of order m^{-4} times a large number, where m is the typical mass scale in the theory. [For example, $\lambda^{-1}\ m^{-4}$, where λ is a coupling constant]. The usual S-matrix calculations apply in the opposite limit of low energy density and since they only provide information on *in* $\rightarrow$ *out* matrix elements, are unsuitable for calculations of expectation values.

Our programme on non-equilibrium dynamics of quantum field theory, started in 1992 [1], is naturally poised to provide a framework to study these problems. Thorough and accurate numerical calculations involving large-N and Hartree nonperturbative methods gave a clear picture of the physical processes involved both, qualitatively and quantitatively [2,7,8]. Very briefly, part of the energy initially stored in the zero mode is dissipated through quantum particle production. Since the initial field amplitude is typically of order $1/\sqrt{\lambda}$, this is a nonperturbative physical process. Detailed analytical calculations yield precise results during the so-called preheating time. Namely, before non-linearities shut-off the process of particle production linked to parametric or spinodal instabilities [8].

We consider here a ϕ^4 scalar field theory in four space-time dimensions (i.e. a standard inflationary model). We give the name of order parameter to the quantum expectation value of the field $\phi(t)$. We consider traslationally invariant quantum states with constant (and finite) energy per unit volume. A nonlinear and nonlocal equation of motion determines the time evolution of $\phi(t)$ [12,1,2]. To one-loop level and to cubic order in $\phi(t)$, the equation of motion can be written as [2]

$$\ddot{\phi}(t) + m^2\phi(t) + \frac{g}{6}\phi^3(t) + \frac{g^2}{4}\phi(t)\int_{t_0}^{t} dt'\ \phi(t')\ \dot{\phi}(t') \int \frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\,\omega_k^3} = 0\ . \tag{1.1}$$

where g and m^2 stand for the renormalized coupling constant and mass and $\omega_k = \sqrt{m^2+k^2}$. Eq.(1.1) is to be supplemented by the initial data $\phi(t_0)$ and $\dot{\phi}(t_0)$.

The purpose of this paper is to solve analytically the equation of motion (1.1) within the renormalization group approach. We obtain in this way analytic expressions for the order parameter at **all times**.

*e-mail:salgado@lpthe.jussieu.fr

†Laboratoire Associé au CNRS UA280.

Our final results can be summarized as follows. The order parameter takes the form

$$\phi(t) = \sqrt{\frac{6m^2}{g}}\,\eta(mt)$$

where

$$\eta(t) = R(t)\;\mathrm{cn}(t\sqrt{1+R(t)^2}, k(t))\;,$$

where cn stands for the Jacobi elliptic cosinus and $R(t)$ and $k(t) \equiv \frac{R(t)}{\sqrt{2(1+R(t)^2)}}$ vary slowly with time. [Slow compared with the variation of the elliptic function whose period is of order one].

The time dependence of $R(t)$ is determined by the renormalization group equations. We give in eq.(7.4) the explicit time dependence.

We find for late times,

$$R(t) - R_\infty = \frac{\lambda\sqrt{\pi}}{32}\,\frac{R_\infty^3}{1-3R_\infty^2}\,\frac{\Omega\cos\Omega t\;\cos(2t+\pi/4)+\sin\Omega t\;\sin(2t+\pi/4)}{(\Omega+2)^2(\Omega-2)\;t^{3/2}} + O(t^{-2})$$

That is, the field amplitude for late times approaches a finite limit as $O(t^{-3/2})$. This limiting value can be expressed as a function of the initial amplitude R_0 by

$$F(R_\infty) - F(R_0) = -\frac{\pi^5\lambda}{\omega_1^2}\left[\frac{\omega_1^2+2}{\omega_1\sqrt{\omega_1^2-1}}\mathrm{Arg}\cosh\omega_1 - 2\right]\;,$$

where the trascendental function $F(R)$ is defined by eqs.(7.6) and (7.8).

For small initial amplitude this reduces to

$$R_\infty \simeq R_0\left(1-\lambda\frac{R_0^2}{8\pi}\right) + \mathcal{O}(\lambda R_0^4)$$

In summary the order parameter oscillates as the classical cnoidal solution with slowly time dependent amplitude and frequency. The amplitude decreases with time due to the energy dissipated in produced particles. These results are in agreement with the numerical calculations [1,2].

II. A PEDAGOGICAL EXAMPLE

The renormalization group method (RG) [15,16,18] improves the knowledge of the asymptotic behaviour of solutions of non-linear differential equations. The long time power law behaviour for non-linear equations can be found by RG methods [15,19]. The RG not only resums the secular terms that arise in an usual (naïve) perturbative expansion but also yields an uniform expansion of the solution. More, it is possible to choose between the different resumed approximations. The different choices are, of course, all equivalent at a given order. We will show that the choice of the final form is given from the choice of the counterterms that are needed in the method. Moreover, after one has choosen the final form of what can be called the envelope of the solution [18], one still has a freedom in the choice of the counterterms and can introduce an arbitrary function expressing a kind of gauge freedom. This gauge function may be used to give a simple expression to the slowly varying pieces. Of course, the final solution is independent of this gauge function.

In order to introduce these ideas we consider the exactly solvable example of the anharmonic oscillator with given initial conditions:

$$\ddot{y} + y + \epsilon y^3 = 0$$
$$y(t_0) = R_0, \quad \dot{y}(t_0) = 0$$

Where the exact solution is given in terms of elliptic functions [13]

$$y(t) = R_0\;\mathrm{cn}\left((t-t_0)\sqrt{1+\epsilon R_0^2}, k(\sqrt{\epsilon}R_0)\right), \qquad k(x) = \frac{x}{\sqrt{2(1+x^2)}} \tag{2.1}$$

Starting from the zeroth order solution (for $\epsilon = 0$)

$$y_0(t) = R_0 \cos(t - t_0)$$

one can easily get the first order solution by linear perturbation (or by expanding eq.(2.1) to order ϵ) and see that it contains a secular term (in $t - t_0$):

$$\begin{aligned} y(t) = &\quad R_0 \; \cos(t + \Theta_0) - \epsilon \; \frac{3}{8}(t - t_0) \; R_0^3 \; \sin(t + \Theta_0) \\ &+ \epsilon \; \frac{R_0^3}{32} \left[\cos 3(t + \Theta_0) - \cos(t + \Theta_0) \right] + \mathcal{O}(\epsilon^2) \\ \Theta(t_0) = &- t_0 \end{aligned} \tag{2.2}$$

Now following the method described in ref. [16] we introduce the renormalization of R_0 and Θ_0:

$$R_0(t_0) = R(\tau) Z_1(t_0, \tau)$$

$$\Theta_0(t_0) = \Theta(\tau) + Z_2(t_0, \tau)$$

Where τ is a constant which we shall choose later, we expand Z_1 and Z_2 in power of ϵ, after we we have noticed that Z_1 is a multiplicative renormalization and that Z_2 is an additive one:

$$Z_1 = 1 + \epsilon a_1 + \mathcal{O}(\epsilon^2)$$

$$Z_2 = \qquad \epsilon b_1 + \mathcal{O}(\epsilon^2)$$

Inserting these expansions in (2.2) we get:

$$\begin{aligned} y(t) = &\quad R \cos(t + \Theta) + \epsilon \; R \; a_1 \cos(t + \Theta) - \epsilon \; R \; b_1 \sin(t + \Theta) \\ &- \epsilon \; \frac{3}{8}(t - \tau) \; R^3 \; \sin(t + \Theta) - \epsilon \; \frac{3}{8}(\tau - t_0) \; R^3 \; \sin(t + \Theta) \\ &+ \epsilon \; \frac{R^3}{32} \left[\cos 3(t + \Theta) - \cos(t + \Theta) \right] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.3}$$

Now we want to remove the term that grows with t_0, because for a fixed (eventually large) t, this is a secular term. Although there are no ultraviolet neither infrared divergences, which are typical of quantum field theory, subtracting such terms is the analogue of eliminating the divergent terms in a quantum field theory calculation ($t_0 \leftrightarrow \ln \Lambda$), where Λ is the ultraviolet cutoff. We call minimal subtraction scheme the one with the following counterterms

$$a_1^{MS} = 0$$

$$b_1^{MS} = \frac{3}{8}(t_0 - \tau)$$

Requiring that $y(t)$ *do not* depend on τ (as bare correlation functions do not depend on the renormalization scale in quantum field theory), we obtain:

$$0 = \frac{dy}{d\tau}(t) = \dot{R}(\tau) \cos(t + \Theta(\tau)) + \epsilon R \left[\frac{3}{8} R(\tau)^2 - \dot{\Theta} \right] \sin\Big(t + \Theta(\tau) \Big) + \mathcal{O}(\epsilon^2)$$

Using the fact that for a fixed τ, $\cos(t + \Theta(\tau))$ and $\sin(t + \Theta(\tau))$ are independent functions; one can see that they are both solution of the zeroth order equation. This yields the renormalization group equations, (which can be obtained using different but equivalent means [17,20]):

$$\begin{aligned} \dot{R} &= 0 + \mathcal{O}(\epsilon^2) \\ \dot{\Theta} &= \epsilon \; \frac{3}{8} R^2 + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.4}$$

Now we consider that R and Θ are solutions of the RG equations, hence $y(t)$ is τ-independent and we can choose τ arbitrarily. We choose $\tau = t$ and we recover the same result as expanding the exact elliptic solution (see ref. [13,21]). (This approximation can be derived by other methods, see ref. [14]):

$$
\begin{aligned}
y(t) &= R_0 \cos\omega(t-t_0) + \epsilon\,\frac{R_0^3}{32}\left[\cos 3\omega(t-t_0) - \cos\omega(t-t_0)\right] + \mathcal{O}(\epsilon^2) \\
\omega &= 1 + \epsilon\,\frac{3}{8}R_0^2 + \mathcal{O}(\epsilon^2)
\end{aligned}
\tag{2.5}
$$

Here the secular term have been resumed [compare with eq.(2.2)].
We could have chosen other counter-terms, for instance:

$$
\begin{aligned}
a_1 &= a_1^{MS} - \epsilon\,\frac{R^2}{32}\left[-1 + \cos 2(\tau+\Theta) - g(\tau+\Theta)\sin(\tau+\Theta)\right] \\
b_1 &= b_1^{MS} - \epsilon\,\frac{R^2}{32}\left[\sin 2(\tau+\Theta) - g(\tau+\Theta)\cos(\tau+\Theta)\right]
\end{aligned}
$$

where $g(u)$ is an arbitrary function. One has to notice that the terms in $\sin 2u$ and $\cos 2u$ are introduced to remove the term in $\cos 3(t+\Theta_0)$ in the first order solution (2.2).
When we insert these values in (2.3) we get:

$$
\begin{aligned}
y(t) = \quad & R\,\cos(t+\Theta) + \epsilon\,\frac{3}{8}\,(\tau - t)R^3\sin(t+\Theta) \\
+\,\epsilon\,\frac{R^3}{32}\Big[& \cos 3(t+\Theta) - \cos(t+\Theta)\cos 2(\tau+\Theta) + \sin(t+\Theta)\sin 2(\tau+\Theta) + \\
& g(\tau+\Theta)(\sin(\tau+\Theta)\cos(t+\Theta) - \cos(\tau+\Theta)\sin(t+\Theta)\Big] + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

The RG equations take now the form:

$$
\begin{aligned}
\dot{R} &= -\epsilon\,\frac{R^3}{32}\left\{2\sin 2(\tau+\Theta) + \frac{d}{d\tau}\left[g(\tau+\Theta)\sin(\tau+\Theta)\right]\right\} + \mathcal{O}(\epsilon^2) \\
\dot{\Theta} &= \epsilon\,\frac{3}{8}R^2 + \epsilon\frac{R^2}{32}\left\{2\cos 2(\tau+\Theta) - \frac{d}{d\tau}\left[g(\tau+\Theta)\cos(\tau+\Theta)\right]\right\} + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

If $R(t)$ and $\Theta(t)$ are solutions of these last RG equations, $y(t)$ is τ-independent, but might depend on $g(u)$. In fact this is not the case, if one puts $\tau = t$ and expands the final solution to see how it depends on $g(u)$ one notices that at this order all the terms containing $g(u)$ cancel. In other words, the final solution **do not** depend on the choice of this function $g(u)$. The arbitrariness of the function $g(u)$ is like a gauge freedom in the present context. We can use this gauge freedom to simplify the RG equations.
In the present case a good choice is:

$$
g(u) = -2\sin u
$$

With this value of the gauge function the RG equations become:

$$
\begin{aligned}
\dot{R} &= 0 + \mathcal{O}(\epsilon^2) \\
\dot{\Theta} &= \epsilon\,\frac{3}{8}R^2 + \epsilon\,\frac{R^2}{8}\cos 2(\tau+\Theta) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

It is clear that if we again choose $\tau = t$, all terms will vanish except the first one:

$$
y(t) = R(t)\cos(t + \Theta(t)) + \mathcal{O}(\epsilon^2)
$$

We finally get:

$$
\begin{aligned}
y(t) &= R_0\cos\left[\omega(t-t_0) + \epsilon\frac{R^2}{16}\sin 2\omega(t-t_0)\right] + \mathcal{O}(\epsilon^2) \\
\omega &= 1 + \epsilon\frac{3}{8}R_0^2 + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Expanding this last result in powers of ϵ we recover the previous equation (2.5).

III. OUT OF EQUILIBRIUM QUANTUM FIELD THEORY

Let us consider the self-coupled ϕ^4 scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}m^2\Phi^2 - \frac{g}{4!}\Phi^4 \tag{3.1}$$

The time evolution of the order parameter defined as the quantum expectation value,

$$\phi(t) = \langle \Phi(x,t) \rangle \tag{3.2}$$

is governed by non-linear and non-local equations. We consider translational invariant quantum states.
As shown in ref. [2], to order g (one-loop level) and keeping the first non-linear contribution in the field amplitude $\phi(t)$, the equation of motion for $\phi(t)$ takes the form:

$$\ddot{\phi}(t) + m^2\phi(t) + \frac{g}{6}\phi^3(t) + \frac{g^2}{4}\phi(t)\int_{t_0}^{t} dt'\; \phi(t')\;\dot{\phi}(t') \int \frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\,\omega_k^3} = 0\;. \tag{3.3}$$

Here g and m^2 stand for the renormalized coupling constant and mass, respectively. We have chosen a finite time t_0 as the beginning of the self-coupling interaction as in ref. [2].

It must be noticed that the last term in eq. (3.3), (the 'dissipative' contribution), has a non-Markovian (i.e. memory-retaining) kernel. Secondly, the equation is *non-linear* in the field amplitude $\phi(t)$. Moreover, as shown in [2], a $\Gamma\dot{\phi}$ term in the equation of motion **cannot** reproduce the dynamics.

Now, we introduce dimensionless variables and replace t by t/m

$$\eta(t) = \sqrt{\frac{g}{6m^2}}\;\phi(\frac{t}{m}) \;\;;\;\; q = \frac{k}{m} \;\;;\;\; \lambda = \frac{3}{8\pi}\,g$$

and eq.(3.3) becomes

$$\ddot{\eta} + \eta + \eta^3 + \lambda\, h[\eta] = 0\;, \tag{3.4}$$

where

$$\begin{aligned} h[\eta(t)] &= \eta(t)\int_{t_0}^{t} \dot{\eta}(t')\eta(t)K(t'-t)dt' \\ K(t) &= \int_0^\infty \frac{k^2dk}{2\pi}\frac{\cos 2\omega t}{\omega^3} \quad \text{and} \quad \omega^2 \equiv 1+k^2 \end{aligned} \tag{3.5}$$

Since t_0 is the beginning of the out of equilibrium evolution the initial conditions for $t = t_0$ are given as in ref. [2]

$$\eta(t_0) = R_0 \quad \text{and} \quad \dot{\eta}(t_0) = 0$$

In order to understand the late time behaviour of the order parameter, we shall solve this non-linear equation (3.4). Using the renormalization groups methods [15,16,18] we will derive this behaviour in term of the initial conditions, up to the first order in λ and third order in the amplitude.

IV. THE NAïVE PERTURBATIVE EXPANSION

We start by seeking a standard perturbative solution in powers of λ,

$$\begin{aligned} &\eta(t) = \eta_0(t) + \lambda\,\eta_1(t) + \mathcal{O}(\lambda^2)\;, \\ &\text{with} \quad \eta_0(t_0) = R_0 \quad \text{and} \quad \dot{\eta}_0(t_0) = 0\;. \end{aligned} \tag{4.1}$$

We have then to solve to zeroth order:

$$\ddot{\eta}_0 + \eta_0 + \eta_0^3 = 0\;,$$

$$\eta_0(t_0) = R_0 \quad \text{and} \quad \dot{\eta}_0(t_0) = 0 \ . \tag{4.2}$$

The first order correction obeys,

$$\ddot{\eta}_1 + \eta_1 + 3\eta_0^2\eta_1 + h[\eta_0] = 0 \ ,$$

$$\eta_1(t_0) = 0 \quad \text{and} \quad \dot{\eta}_1(t_0) = 0 \ . \tag{4.3}$$

We now introduce the function:

$$f(t,R) = R \ \mathrm{cn}(t\sqrt{1+R^2}, k(R)) \ ,$$

with

$$k(R) = \frac{R}{\sqrt{2(1+R^2)}} \ . \tag{4.4}$$

It is straightforward to show [13] that $f(t,R)$ obeys eq.(4.2) for $t_0 = 0$.

$$\ddot{f} + f + f^3 = 0 \ ,$$

$$f(0) = R \quad , \quad \dot{f}(0) = 0 \ .$$

So we have to zeroth-order the (purely classical) solution:

$$\eta_0(t) = f(t - t_0, R_0) \ . \tag{4.5}$$

In order to find the one-loop quantum correction η_1, we have to solve the linear differential equation (4.3). We define:

$$f_1(t,R) \equiv \frac{\partial f}{\partial t}(t,R) \quad , \quad f_2(t,R) \equiv \frac{\partial f}{\partial R}(t,R) \ .$$

As $f(t)$ is a solution of (4.2) and $f_1(t)$ and $f_2(t)$ are both derivatives of $f(t)$, $f_1(t)$ and $f_2(t)$ are solutions of the homogeneous part of (4.3). Using the Green's function method, we can write the solution of eq.(4.3) as

$$\eta_1(t) = \frac{1}{W_0}\left[f_2(t-t_0)\int_{t_0}^{t} f_1(t'-t_0)h(t')dt' - f_1(t-t_0)\int_{t_0}^{t} f_2(t'-t_0)h(t')dt'\right] \ . \tag{4.6}$$

Where $h(t)$ stands for $h[\eta_0(t)]$ and where W_0 is the Wronskian of $f_1(t,R)$ and $f_2(t,R)$:

$$W_0 \equiv W[f_1(t,R), f_2(t,R)] = -R_0(1+R_0^2) \ .$$

We now have the first two terms of the perturbative expansion of $\eta(t)$ in powers of λ . Unfortunately, they contain **secular** terms that make the expansion (4.1) useless for $t \geq \lambda^{-1}$. To see this, one can Fourier expand the periodic functions $f(t)$ and $f_1(t)$ (see [13]) as:

$$\begin{aligned} f(t,R) &= \sum_{n=0}^{\infty} f_n \ \cos(2n-1)\Omega t \\ f_1(t,R) &= -\Omega\sum_{n=0}^{\infty}(2n-1) \ f_n \ \sin(2n-1)\Omega t \\ \Omega \equiv \frac{\pi}{K(k)}\sqrt{1+R^2}, &\qquad f_n \equiv \frac{2\pi R}{k \ K(k)}\frac{q^{n-1/2}}{1+q^{2n-1}} \ , \qquad q \equiv e^{-\pi K'(k)/K(k)} \ . \end{aligned} \tag{4.7}$$

One can notice that the function $f_2(t)$ is not periodic in t. However, the function $f_3(t)$ defined as

$$f_3(t) \equiv f_2(t) - t\ \frac{\Omega'}{\Omega} f_1(t) \ ,$$

is a periodic function of t with the Fourier expansion

$$f_3(t) = \sum_{n=0}^{\infty} f'_n \cos(2n-1)\Omega t \ .$$

Here $'$ stands for $\frac{\partial}{\partial R}$.

This shows that $f_2(t)$ contains a secular term, that will contribute through eq.(4.6), to the secular terms in $\eta_1(t)$. The integrals of $f_1(t-t')h(t')$ and $f_2(t-t')h(t')$ in eq.(4.6) yield secular terms. The second expression produces a second order secular term (growing as t^2).

V. RENORMALIZATION GROUP EQUATIONS

In order to analyze the late time behaviour of the parameter, we want to recast $\eta(t)$ in the form:

$$\eta(t) = f(t + \Theta(t), R(t)) \ ,$$

where the t dependence of Θ and R is slow. That is, $\dot\Theta = \mathcal{O}(\lambda)$, $\dot R = \mathcal{O}(\lambda)$. Since $f(t)$ is an amplitude R multiplied by an oscillatory function (the elliptic cosine), $R(t)$ will be the envelope of the solution.

In order to do that, we write the parameters R_0 and $\Theta_0(t_0) \equiv -t_0$ in terms of new parameters Θ and R through (finite) renormalizations:

$$R_0(t_0) = R(\tau) \ Z_1(t_0, \tau)$$

$$\Theta_0(t_0) = \Theta(\tau) + Z_2(t_0, \tau)$$

Here τ is a (new) constant parameter and Z_1 and Z_2 have the following expansion in power of λ, which ensure that both R and Θ are of order $\mathcal{O}(\lambda)$:

$$Z_1 = 1 + \lambda\, a_1(t_0, \tau) + \mathcal{O}(\lambda^2)$$

$$Z_2 = 0 + \lambda\, b_1(t_0, \tau) + \mathcal{O}(\lambda^2)$$

The requirement that $\eta(t) = \eta_0(t) + \lambda\, \eta_1(t) + \mathcal{O}(\lambda^2)$ takes the form $f(t + \Theta(\tau), R(\tau))$ up to $\mathcal{O}(\lambda^2)$ fixes the (finite) counterterms a_1 and b_1 up to an arbitrary 'gauge' function $g(\tau)$. We find using eqs.(4.5) and (4.6),

$$\begin{aligned} a_1(t_0,\tau) &= \frac{g(\tau)}{W} f_1(\tau+\Theta) - \frac{1}{W}\int_{t_0}^{\tau} f_1(t'+\Theta)h(t')dt' \quad , \\ b_1(t_0,\tau) &= -\frac{g(\tau)}{W} f_3(\tau+\Theta) - a_1(t_0,\tau)\,(\tau-t_0)\frac{\Omega'}{\Omega} \\ &\quad -\frac{1}{W}\left[\frac{\Omega'}{\Omega}(\tau-t_0)\int_{t_0}^{\tau} f_1(t'+\Theta)h(t')dt' - \int_{t_0}^{\tau} f_2(t'+\Theta)h(t')dt'\right] \quad . \end{aligned}$$

Where we have written Θ for $\Theta(\tau)$, R for $R(\tau)$ and $f_i(t)$ for $f_i(t, R(\tau))$. We find,

$$\begin{aligned} \eta(t) &= f(t+\Theta,R) - \lambda\,\frac{g(\tau)}{W}\left[f_1(t+\Theta)f_3(\tau+\Theta) - f_3(t+\Theta)f_1(\tau+\Theta)\right] \\ &\quad +\lambda\,\frac{g(\tau)}{W} f_1(\tau+\Theta)f_1(t+\Theta)\frac{\Omega'}{\Omega}(t-\tau) \\ &\quad +\frac{\lambda}{W}\left\{\left[f_3(t+\Theta) + (t-t_0)\frac{\Omega'}{\Omega}f_1(t+\Theta)\right]\int_{\tau}^{t} f_1(t'+\Theta)h(t')dt' \right. \\ &\quad \left. -f_1(t+\Theta)\int_{\tau}^{t} f_2(t'+\Theta)h(t')dt'\right\} + \mathcal{O}(\lambda^2) \ . \end{aligned} \tag{5.1}$$

Since $\eta(t)$ is clearly τ independent:

$$\frac{d\eta}{d\tau}(t) = 0 \quad , \qquad \forall t \quad . \tag{5.2}$$

Imposing eq.(5.2) to eq.(5.1) yields the renormalization group equations to order λ.

$$\begin{aligned} \dot R(t) &= \ \frac{\lambda}{W}\left\{h(t)f_1(t+\Theta(t)) - \frac{d}{dt}\Big[g(t)f_1(t+\Theta(t))\Big]\right\} \\ \dot\Theta(t) &= -\frac{\lambda}{W}\left\{h(t)f_2(t+\Theta(t)) - \frac{d}{dt}\Big[g(t)f_2(t+\Theta(t))\Big]\right\} \quad . \end{aligned} \tag{5.3}$$

In the calculations, we used the fact that $\dot\Theta$ and $\dot R$ are of order $\mathcal{O}(\lambda)$ and the linear independence of f_1 and f_2.

Since τ is arbitrary and $\eta(t)$ is τ independent we can set τ equal to t in eq.(5.1). We thus find,

$$\eta(t) = f(t + \Theta(t), R(t)) + \mathcal{O}(\lambda^2) \ .$$

We have thus succeeded to resum the secular terms.

VI. FIXING THE 'GAUGE' FUNCTION

We introduced an arbitrary function $g(\tau)$ in the counter terms a_1 and b_1. This function appears in the renormalization group equations. The solution $\eta(t)$ must be independent of the choice of $g(\tau)$. This is easy to prove to this order. One can choose the freedom in g to give to Θ or R a specific form. It is clear that $\dot{\Theta}$ is related to the finite mass renormalization and so has to be bounded for large t.

We now study $\dot{\Theta}$ from eq.(5.3). We have to compute

$$h(\tau) = f(\tau+\Theta) \int_0^\tau dt' \ f\dot{f}(t'+\Theta) \int \ d\mu(\omega) \ \cos 2\omega(t'-\tau) \ .$$

Where the integration measure for ω is

$$d\mu(\omega) = \frac{k^2}{2\pi\omega^3}\, dk = \frac{\sqrt{\omega^2-1}}{2\pi\omega^2}\, d\omega$$

We introduce

$$I(t) = \int_0^t dt' \ f\dot{f}(t'+\Theta) \int \ d\mu(\omega) \ \cos 2\omega(t'-t) \ ,$$

such that $h(\tau) = f(\tau+\Theta)\ I(\tau)$.

Using the expansion of the Jacobian elliptic function (see [21]) and the notations (4.7) we have

$$f^2(t) = R^2 \left[\frac{E(k)}{K(k)} - k'^2 + \frac{2\pi^2}{k^2 K^2} \sum_{n=1}^{\infty} \frac{n q^n}{1-q^{2n}} \cos n\Omega t \right] \tag{6.1}$$

and

$$f f_1(t+\Theta) = -2\Omega^3 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} \sin n\Omega(t+\Theta) \tag{6.2}$$

where $K(k)$ and $E(k)$ stand for complete elliptic integrals of first and second kind, respectively.

Then it is straightforward to obtain that:

$$\begin{aligned} I(t) = \Omega^3 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} \int d\mu(\omega) \ & \left[\frac{\cos n\Omega(t+\Theta) - \cos(n\Omega\Theta - 2\omega t)}{2\omega + n\Omega} \right. \\ & \left. + \ \frac{\cos(2\omega t + n\Omega\Theta) - \cos n\Omega(t+\Theta)}{2\omega - n\Omega} \right] \end{aligned} \tag{6.3}$$

We study the behaviour at large t of each term of $I(t)$. For the first integral we trivially get

$$\begin{aligned} \int d\mu(\omega) \ \frac{\cos n\Omega(t+\Theta)}{2\omega+n\Omega} &= c_n^{(1)} \cos n\Omega(t+\Theta) \\ c_n^{(1)} &= \int_1^\infty \frac{\sqrt{\omega^2-1}}{2\pi\omega^2} \frac{1}{2\omega+n\Omega}\, d\omega \end{aligned}$$

We find for the second integral

$$-\int d\mu(\omega) \frac{\cos(n\Omega\Theta - 2\omega t)}{2\omega+n\Omega} \overset{t\to\infty}{=} \frac{\sqrt{\pi}}{8} \frac{\cos(2t - n\Omega\Theta + \pi/4)}{(2+n\Omega)t^{3/2}} + \mathcal{O}(\frac{1}{t^{5/2}}) \tag{6.4}$$

The last integral can be written as

$$\int d\mu(\omega) \frac{\cos(2\omega t + n\Omega\Theta) - \cos n\Omega(t+\Theta)}{2\omega - n\Omega} = c_n^{(2)}(t) \ \cos n\Omega(t+\Theta) - s_n^{(2)}(t) \ \sin n\Omega(t+\Theta)$$

Where,

$$
\begin{aligned}
c_n^{(2)}(t) &\overset{t\to\infty}{=} -\text{v.p.}\int_1^\infty \frac{\rho(\omega)}{2(\omega-\omega_n)}d\omega + \mathcal{O}(\frac{1}{t^2}) \\
s_n^{(2)}(t) &\overset{t\to\infty}{=} \frac{\pi}{2}\rho(\omega_n) + \mathcal{O}(\frac{1}{t^2})
\end{aligned}
$$

Here $\rho(\omega)$ and ω_n are defined as:

$$
\rho(\omega) = \frac{\sqrt{\omega^2-1}}{2\pi\omega^2} \quad , \quad \omega_n = \frac{n\Omega}{2} \ .
$$

We now write $I(t)$ as

$$
I(t) = I_{reg}(t) + I_{div}(t) \ .
$$

$I_{div}(t)$ gives for large t unbounded contributions to Θ while $I_{reg}(t)$ only gives regular terms

$$
\begin{aligned}
I_{div}(t) &= \Omega^3 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} \left[C_n \cos n\Omega(t+\Theta) - S_n \sin n\Omega(t+\Theta)\right] \ , \\
C_n &= \frac{1}{2}\ \text{v.p.}\int_1^\infty \left(\frac{1}{\omega+\omega_n} - \frac{1}{\omega-\omega_n}\right) d\mu(\omega), \\
S_n &= \frac{\pi}{2}\rho(\omega_n) \ .
\end{aligned} \tag{6.5}
$$

We rewrite the renormalization group equations using f_1, f_2, f_3, I_{div} and I_{reg} as:

$$
\begin{aligned}
\dot\Theta(t) &= -\frac{\lambda}{W}\Bigg\{ I_{div}(t)\, f f_3(t+\Theta) + I_{reg}(t)\, f f_2(t+\Theta) \\
&\quad - \frac{d}{dt}\Big[g(t) f_3(t+\Theta)\Big] - \frac{\Omega'}{\Omega}\, g(t)\, f_1(t+\Theta) \\
&\quad + \frac{\Omega'}{\Omega}\, t\Big[I_{div}(t)\, f f_1(t+\Theta) - \frac{d}{dt}\Big(g(t) f_1(t+\Theta)\Big)\Big]\Bigg\} \ , \\
\dot R(t) &= \frac{\lambda}{W}\Bigg\{ I_{reg}(t)\, f f_1(t+\Theta) + I_{div}(t)\, f f_1(t+\Theta) - \frac{d}{dt}\Big[g(t) f_1(t+\Theta)\Big]\Bigg\} \ .
\end{aligned}
$$

Since the final resumed solution do not depend on $g(t)$ and since $\eta(t)$ is finite we may choose $g(t)$, in order to find a simple expression for $R(t)$, as a solution of the equation

$$
I_{div}(t) f f_1(t+\Theta) = \frac{d}{dt}\left[g(t) f_1(t+\Theta)\right]
$$

That is,

$$
g(t) = \frac{1}{f_1(t+\Theta)} \int_{t_0}^{t} I_{div}(t') f f_1(t'+\Theta)\, dt' \ .
$$

Now the renormalization group equations become

$$
\begin{aligned}
\dot\Theta(t) &= -\frac{\lambda}{W}\Bigg\{ I_{div}(t)\, f f_3(t+\Theta) + I_{reg}(t)\, f f_2(t+\Theta) \\
&\quad - \frac{d}{dt}\Big[g(t) f_3(t+\Theta)\Big] - \frac{\Omega'}{\Omega}\, g(t) f_1(t+\Theta)\Bigg\} \\
\dot R(t) &= \frac{\lambda}{W}\, I_{reg}(t)\, f f_1(t+\Theta) \ .
\end{aligned} \tag{6.6}
$$

VII. THE ORDER PARAMETER BEHAVIOUR FROM THE RENORMALIZATION GROUP EQUATIONS

We have now the renormalized group equations (6.6) and we must remember that the field equations of motion (3.4) are valid up to $\mathcal{O}(R_0^3)$. We have, using the expansion (6.2) and (6.5) in eq.(6.6)

$$
\begin{aligned}
I_{div}(t) &= \Omega^3 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} \left[C_n \cos n\Omega(t+\Theta) - S_n \sin n\Omega(t+\Theta) \right] \\
I_{reg}(t) &= I(t) - I_{div}(t)
\end{aligned}
\tag{7.1}
$$

We now introduce the following notations:

$$
\alpha_n = \frac{1}{q} \frac{n^2 q^n}{1-q^{2n}} \quad , \quad F'(R) = \frac{K(R)^6}{q(R)^2} \frac{R}{(1+R^2)^2}
\tag{7.2}
$$

With this notations the renormalization group equation (6.6) for $R(t)$ can be written as:

$$
\begin{aligned}
F'(R)\,\dot{R} = \frac{\pi^6 \lambda}{2} \sum_{n,m=1}^{\infty} \alpha_n \alpha_m \Bigg\{ & \pi\rho(\omega_n) \Big(\cos 2(\omega_m - \omega_n)t - \cos 2(\omega_m + \omega_n)t \Big) \\
& + \text{reg} \int_1^{\infty} \left(\frac{1}{\omega - \omega_n} - \frac{1}{\omega + \omega_n} \right) \Big(\sin 2(\omega_m + \omega)t - \sin 2(\omega - \omega_m)t \Big) d\mu(\omega) \Bigg\}
\end{aligned}
$$

Where where by 'reg' we mean the integral regularized by subtracting the appropriate counterterm contained in C_n [see eq.(6.5)]. We used eqs.(6.2-6.3) and (7.1) and we have set $t_0 = 0$ since the eq.(3.4) is invariant under time translations. Moreover, at this order of λ we can set $\Theta = 0$. Integrating this last expression with respect to t yields:

$$
\begin{aligned}
F(R(t)) - F(R_0) = \frac{\lambda \pi^6}{4} \sum_{n,m=1}^{\infty} \alpha_n \alpha_m \Bigg\{ & \pi\rho(\omega_n) \left(\frac{\sin 2(\omega_n - \omega_m)t}{\omega_n - \omega_m} - \frac{\sin 2(\omega_m + \omega_n)t}{\omega_n + \omega_m} \right) \\
& + \text{reg} \int_1^{\infty} d\mu(\omega) \left(\frac{1}{\omega - \omega_n} - \frac{1}{\omega + \omega_n} \right) \left(\frac{1 - \cos 2(\omega + \omega_m)t}{\omega + \omega_m} - \frac{1 - \cos 2(\omega - \omega_m)t}{\omega - \omega_m} \right) \Bigg\}
\end{aligned}
\tag{7.3}
$$

The terms which look singular for $n = m$ must be understood in their limiting value $n \to m$, which is always regular.

One should notice that, unlike the other terms of the integral, the last term gives (for n equal to m) a secular term, which is of course canceled by the last first $2\pi\rho(\omega_n)\, t$:

$$
\int_1^{\infty} d\mu(\omega) \frac{1 - \cos 2(\omega - \omega_n)t}{(\omega - \omega_n)^2} - 2\pi\rho(\omega_n) t \overset{t \to \infty}{=} \text{v.p.} \int_1^{\infty} \frac{d\mu(\omega)}{(\omega - \omega_n)^2} + \mathcal{O}(\frac{1}{t^2})
$$

The equations of motion (3.4)-(3.5) are valid to order R_0^3 and $q = O(k^2) = O(R_0^2)$ for small R_0. Therefore, we can keep the terms in the sum (7.3) where n and m are equal to 1. We find from eq.(7.3) after non-trivial cancelations,

$$
\begin{aligned}
F(R(t)) - F(R_0) = \frac{\lambda \pi^6}{4} \Bigg\{ & \pi\rho(\omega_1) \left(2t - \frac{\sin 4\omega_1 t}{2\omega_1} \right) \\
& + \text{reg} \int_1^{\infty} d\mu(\omega) \left(\frac{1}{\omega - \omega_1} - \frac{1}{\omega + \omega_1} \right) \left(\frac{1 - \cos 2(\omega + \omega_1)t}{\omega + \omega_1} - \frac{1 - \cos 2(\omega - \omega_1)t}{\omega - \omega_1} \right) , \Bigg\}
\end{aligned}
\tag{7.4}
$$

where $\omega_1 = \Omega/2 = \frac{\pi}{2K(k)} \sqrt{1+R^2}$.

In order to find the field amplitude for $t \to \infty$, we average over an oscillation period

$$
F(R_\infty) - F(R_0) = -\frac{\pi^5 \lambda}{8} \text{ v.p.} \int_1^{\infty} d\omega \, \frac{\sqrt{\omega^2 - 1}}{\omega^2} \left[\frac{1}{(\omega + \omega_1)^2} + \frac{1}{(\omega - \omega_1)^2} \right]
$$

The last integral gives:

$$F(R_\infty) - F(R_0) = -\frac{\pi^5 \lambda}{\omega_1^2} \left[\frac{\omega_1^2 + 2}{\omega_1 \sqrt{\omega_1^2 - 1}} \mathrm{Arg}\cosh \omega_1 - 2 \right] \; , \tag{7.5}$$

One can easily see that the expression between the brackets is positive. Since $F'(R)$ is positive R_∞ is smaller than R_0. This result was expected since we have a dissipative behaviour. Namely, part of the energy initially in the zero mode is spent in particle production.

More explicitly,

$$F(R) \equiv \int_\infty^R \frac{K(k(R))^6}{q(k(R))^2} \frac{R\, dR}{(1+R^2)^2} = \int_{1/\sqrt{2}}^{k(R)} k \; dk \; \frac{K(k)^6}{q(k)^2} \; . \tag{7.6}$$

where $k(R)$ is defined by eq.(4.4).

For late t, $R(t) - R_\infty$ is small and we can write

$$F(R(t)) - F(R_\infty) = F'(R_\infty)(R(t) - R_\infty) \; .$$

We can then evaluate $F(R(t))$ for late t from eq.(7.4) using eq.(6.4),

$$F(R(t)) - F(R_\infty) = \pi^6 \; \lambda \frac{\sqrt{\pi}}{4} \int_t^\infty \frac{\cos(2t' + \pi/4) \sin \Omega t'}{(2+\Omega) t'^{3/2}} \, dt'$$

We find equating both results the behaviour of $R(t)$ for late t:

$$R(t) - R_\infty = \frac{\lambda \sqrt{\pi}}{32} \frac{R_\infty^3}{1 - 3R_\infty^2} \frac{\Omega \cos \Omega t \; \cos(2t + \pi/4) + \sin \Omega t \; \sin(2t + \pi/4)}{(\Omega+2)^2 (\Omega - 2) \; t^{3/2}} + O(t^{-2}) \tag{7.7}$$

Eqs.(7.5) and (7.6) define R_∞ as a function of R_0 and λ.

In the present case it is very convenient to express the elliptic functions in powers of the elliptic nome $q = e^{-\pi K'/K}$ since q is always small in our context. Namely, $0 \leq q \leq e^{-\pi} = 0.0432139\ldots$ for $0 \leq R_0 \leq \infty$. We have for example,

$$K(k) = \frac{\pi}{2} \left[1 + 4q + 4q^2 + \mathcal{O}(q^4) \right]$$
$$k^2 = 16 q \left[1 - 8q + 44 q^2 + \mathcal{O}(q^3) \right] \; .$$

The function $F(R)$ becomes

$$F(R) = -\frac{\pi^6}{8} \; \left[\frac{1}{q} - 8 \log q - 12 q + \mathcal{O}(q^2) \right] + \mathcal{C} \; . \tag{7.8}$$

where the constant $\mathcal{C}$ is given by $\mathcal{C} = \frac{\pi^6}{8} \; e^\pi \; \left[1 + 8\pi e^{-\pi} - 12 e^{-2\pi} + \mathcal{O}(e^{-3\pi}) \right] = 5738.9\ldots$

Eq.(7.5) and (7.8) indicates us that $q_\infty \equiv q(R_\infty)$ and $q_0 \equiv q(R_0)$ are related as follows,

$$q_\infty - q_0 = -c \, \lambda + \mathcal{O}(\lambda^2) \; .$$

where,

$$c = \frac{8 \, q_0^2}{\pi \omega_1^2} \left[\frac{\omega_1^2 + 2}{\omega_1 \sqrt{\omega_1^2 - 1}} \mathrm{Arg}\cosh \omega_1 - 2 \right] [1 - 8 q_0 + \mathcal{O}(q_0^2)] \; .$$

More explicitly, for $R_0 \to 0$ we find:

$$c \simeq \frac{8}{\pi} q_0^2 \simeq \frac{R_0^4}{128 \pi} + \mathcal{O}(R_0^6)$$

That is:

$$R_\infty \simeq R_0 \left(1 - \lambda \frac{R_0^2}{8\pi} \right) + \mathcal{O}(\lambda R_0^4) \; .$$

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